



Compensation of the Null modes with the Gemini MCAO

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This document present a short study of the null modes for the Gemini MCAO system. First, I briefly describe the problem, then I derive the expression of the combined effect of the quadratic modes from an arbitrary number of layers on the plate scale of the science image, in function of the C_n^2 and the wind profiles and the NGS characteristics. Last, I present some performance evaluations for typical guide star configurations and magnitudes.

1 Introduction and Notations

1.1 The problem

It was demonstrated earlier by B.Ellerbroek and later in this paper that in a MCAO system, because wavefront sensing on LGS is insensitive to each beam's overall Tip-Tilt component, not only tip-tilt can not be sensed but also the altitude of the quadratic modes (defocus and astigmatism) can not be determined. A mismatch of the altitude at which these quadratic modes are compensated will result in a plate scale error in the science image, and in turn in a Strehl ratio reduction in long exposure images. These quadratic modes, badly determined by the LGS high order sensors, are called in this study the "null modes" of the LGS high order loop, as they belong to its null space. To solve that problem, one has therefore to rely on multiple Tip-tilt guide star. Becasue there are 5 null modes (tip, tilt, focus and 2 astigmatism), a minimum of 3 guide stars are necessary (2 measurements per GS).

1.2 Definitions

In the following, I will assume that the turbulence is made of discrete layers, although the results of this analysis could easily be extended to a continuous profile.

I use the regular Noll definition for Zernike 2 to 6:

$$\begin{aligned} Z_2 &= 2x \\ Z_3 &= 2y \\ Z_4 &= 2\sqrt{3}(x^2 + y^2) - \sqrt{3} \\ Z_5 &= 2\sqrt{6}xy \\ Z_6 &= \sqrt{6}(x^2 - y^2) \end{aligned}$$

$a_n(h)$ is the coefficient of Z_n at altitude h layer. φ_h is the phase at layer of altitude h .

2 Expression of the Tip and Tilt error versus the field angle

The expansion of the phase on Zernike 2 to 6 in the telescope pupil, integrated along a direction (θ_x, θ_y) at time t is:

$$\varphi_{2...6}(x, y, \theta_x, \theta_y, t) = \sum_h \varphi_{2...6\ h}(x + \theta_x h/R, y + \theta_y h/R, t) = \sum_h \sum_{i=2...6} a_i(h) Z_i(x + \theta_x h/R, y + \theta_y h/R, t) \quad (1)$$

I have dropped the index 2...6 in the following for shortness. The Zernike expansion of φ_h can be developped:

$$\begin{aligned} \varphi_h(x + \frac{\theta_x h}{R}, y + \frac{\theta_y h}{R}) &= a_2(h).2(x + \theta_x h/R) + a_3(h).2(y + \theta_y h/R) + \\ &\quad a_4(h).2\sqrt{3} [(x + \theta_x h/R)^2 + (y + \theta_y h/R)^2 - 1/2] + \\ &\quad a_5(h).\sqrt{6} [2(x + \theta_x h/R)(y + \theta_y h/R)] + \\ &\quad a_6(h).\sqrt{6} [(x + \theta_x h/R)^2 - (y + \theta_y h/R)^2] \end{aligned} \quad (2)$$

$$\begin{aligned} &= 2x(a_2(h) + 2\sqrt{3} \theta_x a_4(h)h/R + \sqrt{6} \theta_y a_5(h)h/R + \sqrt{6} \theta_x a_6(h)h/R) + \\ &\quad 2y(a_3(h) + 2\sqrt{3} \theta_y a_4(h)h/R + \sqrt{6} \theta_x a_5(h)h/R - \sqrt{6} \theta_y a_6(h)h/R) \end{aligned} \quad (3)$$

Where all the constant phase terms (piston) have been ommitted, together with the field independant focus and astigmatism terms, which in a MCAO system are compensated by the high order loop, using LGS information. In Eq 3, the coefficients a_2 and a_3 refer to a global, field independant Tip-Tilt. The other terms depends upon θ_x and θ_y and are plate scale variation terms induced by non-zero altitude quadratic modes. Using the short notation

$$A_{2,3} = \sum_h a_{2,3}(h) \quad \text{and} \quad A_{4,5,6} = \sum_h a_{4,5,6}(h).h \quad (4)$$

we have

$$\begin{aligned} \varphi_{2,3}(x, y, \theta_x, \theta_y, t) &= 2x(A_2(t) + 2\sqrt{3} \theta_x/RA_4(t) + \sqrt{6} \theta_y/RA_5(t) + \sqrt{6} \theta_x/RA_6(t)) + \\ &\quad 2y(A_3(t) + 2\sqrt{3} \theta_y/RA_4(t) + \sqrt{6} \theta_x/RA_5(t) - \sqrt{6} \theta_y/RA_6(t)) \end{aligned} \quad (5)$$

Let us now consider a MCAO system with one mirror conjugated at an altitude h_m , capable of producing tip, tilt, focus and astigmatism. φ_M is the phase and α_n is the coefficient of Zernike Z_n produced by this mirror. Excluding again the field independant focus and astigmatism modes, the compensated phase over the null modes can be written:

$$\begin{aligned} \varphi_{c\ 2,3}(x, y, \theta_x, \theta_y, t) &= \varphi(x, y, \theta_x, \theta_y, t) - \varphi_M(x, y, \theta_x, \theta_y, t) \\ &= 2x(A_2(t) + 2\sqrt{3} \theta_x/RA_4(t) + \sqrt{6} \theta_y/RA_5(t) + \sqrt{6} \theta_x/RA_6(t)) + \\ &\quad 2y(A_3(t) + 2\sqrt{3} \theta_y/RA_4(t) + \sqrt{6} \theta_x/RA_5(t) - \sqrt{6} \theta_y/RA_6(t)) \\ &\quad - 2x(\alpha_2(t) + 2\sqrt{3} \theta_x/R\alpha_4(t)h_m + \sqrt{6} \theta_y/R\alpha_5(t)h_m + \sqrt{6} \theta_x/R\alpha_6(t)h_m) \\ &\quad - 2y(\alpha_3(t) + 2\sqrt{3} \theta_y/R\alpha_4(t)h_m + \sqrt{6} \theta_x/R\alpha_5(t)h_m - \sqrt{6} \theta_y/R\alpha_6(t)h_m) \end{aligned} \quad (6)$$

From this equation, it is easy to see that a single mirror, located at a non zero altitude, is able to compensate for the plate scale variations induced by the atmosphere quadratic modes. Note that a pupil-conjugated mirror is needed to compensate for the induced focus and astigmatism modes in the integrated phase.

The obvious and unique solution to $\varphi_c = 0$ is, in presence of noise n

$$\begin{cases} \alpha_{2,3} = \hat{A}_{2,3} = \sum_h a_{2,3}(h) + n_{2,3} \\ \alpha_{4,5,6} = (1/h_m)\hat{A}_{4,5,6} = (1/h_m) [\sum_h a_{4,5,6}(h)h + n_{4,5,6}] \end{cases} \quad (8)$$

Here I have neglected any spatial aliasing contribution, which are deemed to be small as the high order terms are corrected by the MCAO loop using the LGSs. Introducing the close loop transfer functions (TFs), H_n (Noise TF) and H_ϵ (error TF), a more general solution can be written for the Fourier transform of α ($\tilde{\alpha}$)

$$\begin{cases} \tilde{\alpha}_{2,3} = H_{bf} \tilde{A}_{2,3} + H_n \tilde{n}_{2,3} \\ \tilde{\alpha}_{4,5,6} = (1/h_m)(H_{bf} \tilde{A}_{4,5,6} + H_n \tilde{n}_{4,5,6}) \end{cases} \quad (9)$$

Let us now express the error on φ_c in function of the close loop characteristics and the noise. Noting ξ the Tip component of the compensated phase for an given point in the field of view, we have

$$\xi(\theta_x, \theta_y, t) = \frac{1}{S} \int_{\mathcal{P}} \varphi_c(x, y, \theta_x, \theta_y, t) \cdot Z_2 \, dx \, dy \quad (10)$$

$$\begin{aligned} &= A_2(t) + 2\sqrt{3} \theta_x / R A_4(t) + \sqrt{6} \theta_y / R A_5(t) + \sqrt{6} \theta_x / R A_6(t) - \\ &\alpha_2(t) - 2\sqrt{3} \theta_x / R \alpha_4(t) h_m - \sqrt{6} \theta_y / R \alpha_5(t) h_m - \sqrt{6} \theta_x / R \alpha_6(t) h_m \end{aligned} \quad (11)$$

and the variance over time

$$\begin{aligned} \langle \xi^2(\theta_x, \theta_y) \rangle_t &= \int (A_2(t) - \alpha_2(t))^2 + 12 (\theta_x / R)^2 (A_4(t) - \alpha_4(t) h_m)^2 + 6 (\theta_x / R)^2 (A_6(t) - \alpha_6(t) h_m)^2 \\ &+ 6 (\theta_y / R)^2 (A_5(t) - \alpha_5(t) h_m)^2 + 4\sqrt{3} \theta_x h_m / R \alpha_2(t) \alpha_4(t) + 2\sqrt{6} \theta_y h_m / R \alpha_2(t) \alpha_5(t) \\ &+ 2\sqrt{6} \theta_x h_m / R \alpha_2(t) \alpha_6(t) + 4\sqrt{3} \sqrt{6} \theta_x \theta_y h_m^2 / R^2 \alpha_4(t) \alpha_5(t) \\ &+ 4\sqrt{3} \sqrt{6} \theta_x^2 h_m^2 / R^2 \alpha_4(t) \alpha_6(t) + 12 \theta_x \theta_y h_m^2 / R^2 \alpha_5(t) \alpha_6(t) \end{aligned} \quad (12)$$

where I used the zero correlation of the first and second degree Zernike terms in the atmosphere ($\langle a_i a_j \rangle = 0$, for $i = 2 \dots 6$, $j = 2 \dots 6$, $i \neq j$). The α , however, can be correlated through the command if noise is present. Using the Parseval theorem,

$$\langle \xi^2(\theta_x, \theta_y) \rangle_t = \int \xi^2(\theta_x, \theta_y, t) \, dt = \int \tilde{\xi}^2(\theta_x, \theta_y, f) \, df \quad (13)$$

$$\int \tilde{\xi}^2(\theta_x, \theta_y, f) \, df = \int df (\tilde{A}_2(f) - \tilde{\alpha}_2(f))^2 + 12 (\theta_x / R)^2 (\tilde{A}_4(f) - \tilde{\alpha}_4(f) h_m)^2 \dots \quad (14)$$

Remembering the close-loop relation, e.g. $\tilde{\alpha} = \tilde{a} = H_{bf} \tilde{a}$ or $\tilde{a} - \tilde{\alpha} = H_\epsilon \tilde{a}$, one gets

$$\begin{aligned} (\tilde{A}_2(f) - \tilde{\alpha}_2(f))^2 &= (\tilde{A}_2 - (\tilde{A}_2 \cdot H_{bf} + \tilde{n}_2 \cdot H_n))^2 \\ &= \tilde{A}_2^2 H_\epsilon^2(g_2) + \tilde{n}_2^2 H_n^2(g_2) \end{aligned} \quad (15)$$

or

$$\begin{aligned} (\tilde{A}_4(f) - \tilde{\alpha}_4(f) \cdot h_m)^2 &= (\tilde{A}_4 - h_m (\frac{1}{h_m} \tilde{A}_4 \cdot H_{bf} + \tilde{n}_4 \cdot H_n))^2 \\ &= \tilde{A}_4^2 H_\epsilon^2(g_4) + h_m^2 \tilde{n}_4^2 H_n^2(g_4) \end{aligned} \quad (16)$$

which leads to the final expression of $\langle \xi^2 \rangle$:

$$\begin{aligned} \langle \xi^2(\theta_x, \theta_y) \rangle_t &= \int df \tilde{A}_2^2 H_\epsilon^2(g_2) + 12 (\theta_x / R)^2 \tilde{A}_4^2 H_\epsilon^2(g_4) + 6 (\theta_y / R)^2 \tilde{A}_5^2 H_\epsilon^2(g_5) + 6 (\theta_x / R)^2 \tilde{A}_6^2 H_\epsilon^2(g_6) \\ &+ \tilde{n}_2^2 H_n^2(g_2) + 12 \theta_x^2 h_m^2 / R^2 \tilde{n}_4^2 H_n^2(g_4) + 6 \theta_y^2 h_m^2 / R^2 \tilde{n}_5^2 H_n^2(g_5) + 6 \theta_x^2 h_m^2 / R^2 \tilde{n}_6^2 H_n^2(g_6) \\ &+ 4\sqrt{3} \theta_x h_m / R \tilde{n}_2 \tilde{n}_4 H_n(g_2) H_n(g_4) + 2\sqrt{6} \theta_y h_m / R \tilde{n}_2 \tilde{n}_5 H_n(g_2) H_n(g_5) \\ &+ 2\sqrt{6} \theta_x h_m / R \tilde{n}_2 \tilde{n}_6 H_n(g_2) H_n(g_6) + 4\sqrt{3} \sqrt{6} \theta_x \theta_y h_m^2 / R^2 \tilde{n}_4 \tilde{n}_5 H_n(g_4) H_n(g_5) \\ &+ 4\sqrt{3} \sqrt{6} \theta_x^2 h_m^2 / R^2 \tilde{n}_4 \tilde{n}_6 H_n(g_4) H_n(g_6) + 12 \theta_x \theta_y h_m^2 / R^2 \tilde{n}_5 \tilde{n}_6 H_n(g_5) H_n(g_6) \end{aligned} \quad (17)$$

In this equation, the terms in $\tilde{A}_i^2 \cdot H_\epsilon^2$ represents the servo lag error, i.e. the high frequency part of the mode power spectral density that is not or not perfectly compensated due to time lag. The second part, or the terms in $\tilde{n}_i^2 \cdot H_n^2$, represents the effect of noise on the compensation of each of the modes. n_i^2 is the noise on mode i propagated from the measurement to the compensated phase through the reconstructor. H_n represents the filtering by the close-loop.

Similarly, for the Tilt error ζ :

$$\begin{aligned} \langle \zeta^2(\theta_x, \theta_y) \rangle_t = & \int df \tilde{A}_3^2 H_\epsilon^2(g_3) + 12 (\theta_y/R)^2 \tilde{A}_4^2 H_\epsilon^2(g_4) + 6 (\theta_x/R)^2 \tilde{A}_5^2 H_\epsilon^2(g_5) + 6 (\theta_y/R)^2 \tilde{A}_6^2 H_\epsilon^2(g_6) \\ & + \tilde{n}_3^2 H_n^2(g_3) + 12 \theta_y^2 h_m^2 / R^2 \tilde{n}_4^2 H_n^2(g_4) + 6 \theta_x^2 h_m^2 / R^2 \tilde{n}_5^2 H_n^2(g_5) + 6 \theta_y^2 h_m^2 / R^2 \tilde{n}_6^2 H_n^2(g_6) \\ & + 4\sqrt{3} \theta_y h_m / R \tilde{n}_2 \tilde{n}_4 H_n(g_2) H_n(g_4) + 2\sqrt{6} \theta_x h_m / R \tilde{n}_2 \tilde{n}_5 H_n(g_2) H_n(g_5) \\ & - 2\sqrt{6} \theta_y h_m / R \tilde{n}_2 \tilde{n}_6 H_n(g_2) H_n(g_6) + 4\sqrt{3} \sqrt{6} \theta_x \theta_y h_m^2 / R^2 \tilde{n}_4 \tilde{n}_5 H_n(g_4) H_n(g_5) \\ & - 4\sqrt{3} \sqrt{6} \theta_y^2 h_m^2 / R^2 \tilde{n}_4 \tilde{n}_6 H_n(g_4) H_n(g_6) - 12 \theta_x \theta_y h_m^2 / R^2 \tilde{n}_5 \tilde{n}_6 H_n(g_5) H_n(g_6) \end{aligned} \quad (18)$$

3 Performance estimation

I present here the results of performance evaluations and system optimizations derived from the above formulation.

The input quantities are :

- The Cn² and the wind profiles. The Cn² profile is the median profile for Cerro Pachon, as derived from several site survey campaigns by Vernin et al (see e.g. <http://www.gemini.edu/documentation/webdocs/rpt/rpt-ao-g0094-1.ps>). A LSQ fit to the profile leads to a 7 layers model (communicated by M.Chun) with the following weights and altitudes:

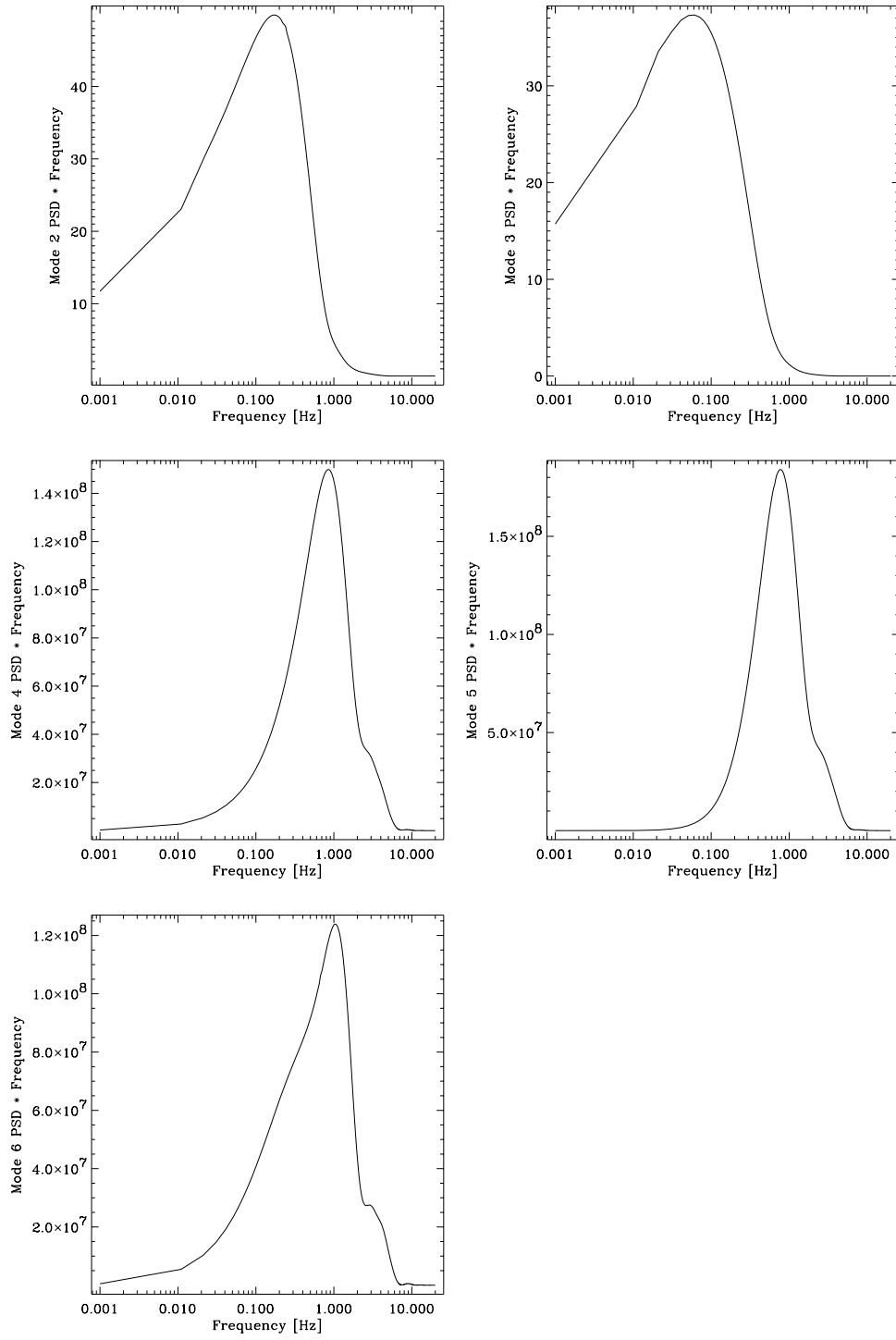
$$\begin{aligned} \text{Cn}^2 [\text{normalized}] &= [0.646, 0.080, 0.119, 0.035, 0.025, 0.080, 0.015] \\ \text{Altitudes [m]} &= [0., 1800, 3300, 5800, 7400, 13100, 15800] \end{aligned}$$

The overall r_0 at 500nm is 0.166 m.

The wind profile has been chosen as follow:

$$\text{Wind [m/s]} = [5., 7.5, 12., 25., 34., 21., 8.]$$

- The null mode temporal Power Spectral Densities (PSDs): For this, I have used Eq 4 and the expression of the $Z_2 \dots Z_6$ given by F.Roddier et al (JOSA A, Vol 10, pp 957–965, 1993). The PSD of the 5 null modes are shown Fig 1. They are plotted as PSD \times frequency versus frequency, for the same reasons as explained in the above mentionned paper (basically, the value of PSD $\times f$ represents the amount of power per unit frequency bin at any given frequency f). The "knee" of the M4...M6 modes at 2-4 Hz is due to the highest layers, which have relatively high velocities and are weighted heavily by their altitude (see Eq 4). The large amplitude of these modes is due to the h^2 weighting term, but is greatly reduced by θ^2 when computing ξ^2 or ζ^2 as per Eq 17 or 18.
- The system close loop transfer functions (TFs): I have used a very basic model for these TFs. It emulates a simple integrator with gain. Examples of H_ϵ and H_n are given in Fig 2 for gain values of 0.1, 0.3 and 0.7.
- The overall noise per mode, i.e. the measurement noise propagated through the reconstructor on each mode. For this I have used a numerical model of the system, which uses average gradient sensors. An

Figure 1: Spectra of the Null mode coefficients A_2 through A_6

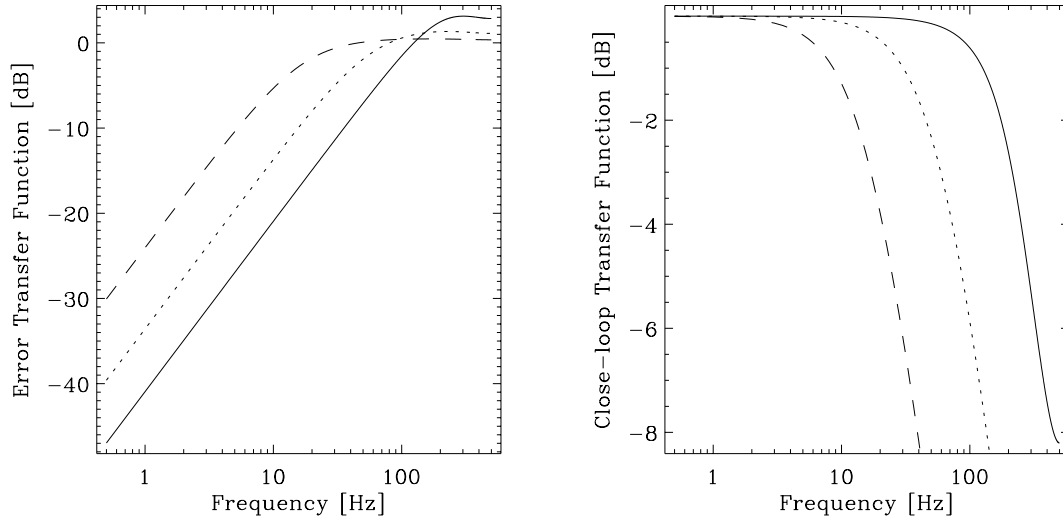


Figure 2: Example of Transfer functions. Delay = 0.1 ms, sampling time = 1ms

interaction matrix D is derived that links image position measurements in milliarcsec to the null mode value in radian (the null modes are normalized so that the variance of the upper mirror is one rd over the telescope pupil area). The noise per mode is then computed using the known relation:

$$|n\rangle = D^{-1}|m\rangle \quad (19)$$

$$|n\rangle\langle n| = D^{-1}|m\rangle\langle m|D^{-1^T} = D^{-1}D^{-1^T} \quad (20)$$

in its more general form ($|n\rangle\langle n|$ is the covariance matrix of the noise on the mode coefficients if $|m\rangle\langle m|$ is the covariance matrix of the noise on the measurements), or

$$|n\rangle\langle n| = D^{-1}D^{-1^T}\sigma_m^2 \rightarrow \langle n_i^2 \rangle = \sigma_m^2 \sum_j D_{ji}^{-1} \cdot D_{ij}^{-1^T} \quad (21)$$

if the noise on all guide star is equal.

The noise on each NGS measurement (X and Y) was expressed as follow (expression provided by BLE):

$$\sigma_m[mas] = 0.587 \frac{(\lambda/r_0)}{SNR} / 4.848e-9 \quad (22)$$

where SNR is the photometric SNR given by :

$$SNR = N_{PDE} / \sqrt{N_{PDE} + 4 N_{SKY}} \quad (23)$$

with

$$\begin{cases} N_{PDE} = 10^{0.4(20-m_*)} \times 5200 / f_{sampling} \\ N_{SKY} = 1300 / f_{sampling} \end{cases} \quad (24)$$

which assumes $m_{sky} = 20 \text{ arcsec}^{-2}$, and a 1 arcsec field stop for the sky reduction. A value of 0.5 has been adopted for the overall optical transmission from the Telescope M1 to the TT sensors, and a QE of 0.6 for the TT sensor detectors (APDs), over a bandwidth of 350 nm. The detector read-out noise (ron) was taken equal to zero (avalanche photodiodes) in these calculations, but it would be straightforward to extend the case to non zero ron detectors by including the appropriate term in Eq 23.

For instance, for a system with 4 NGS at $(\pm 30, \pm 30)$ arcsec from the center, and a DM conjugated at 8000 m, the matrix $D^{-1} \cdot D^{-1^T}$ is diagonal with diagonal elements equal to 0.0038 rd²/mas for M2, M3, 0.0018 rd²/mas for M4 and 0.0038 rd²/mas for M5 and M6.

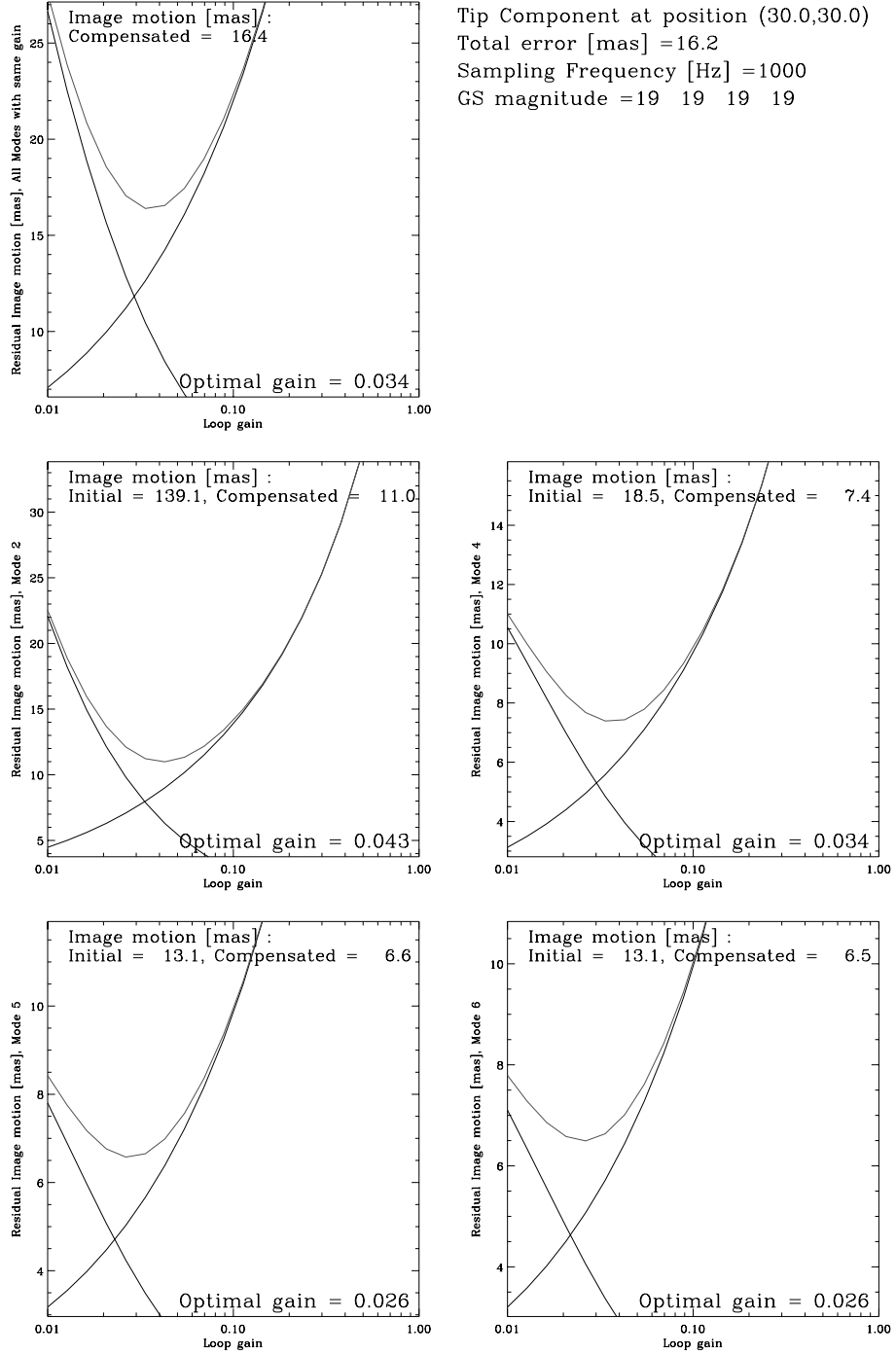


Figure 3: Example of the optimization process for the Tip error. The mode residuals are computed versus the gain of the close-loop, as per Eq. The noise, propagated on the phase through the system control matrix and the close-loop transfer function, is computed for the same gain. The grey curve is the total error versus gain, i.e. the quadratic sum of the residual turbulence and the noise. In this case, four guide stars of magnitude 19 are used at $(\pm 30, \pm 30)$ arcsec off-axis.

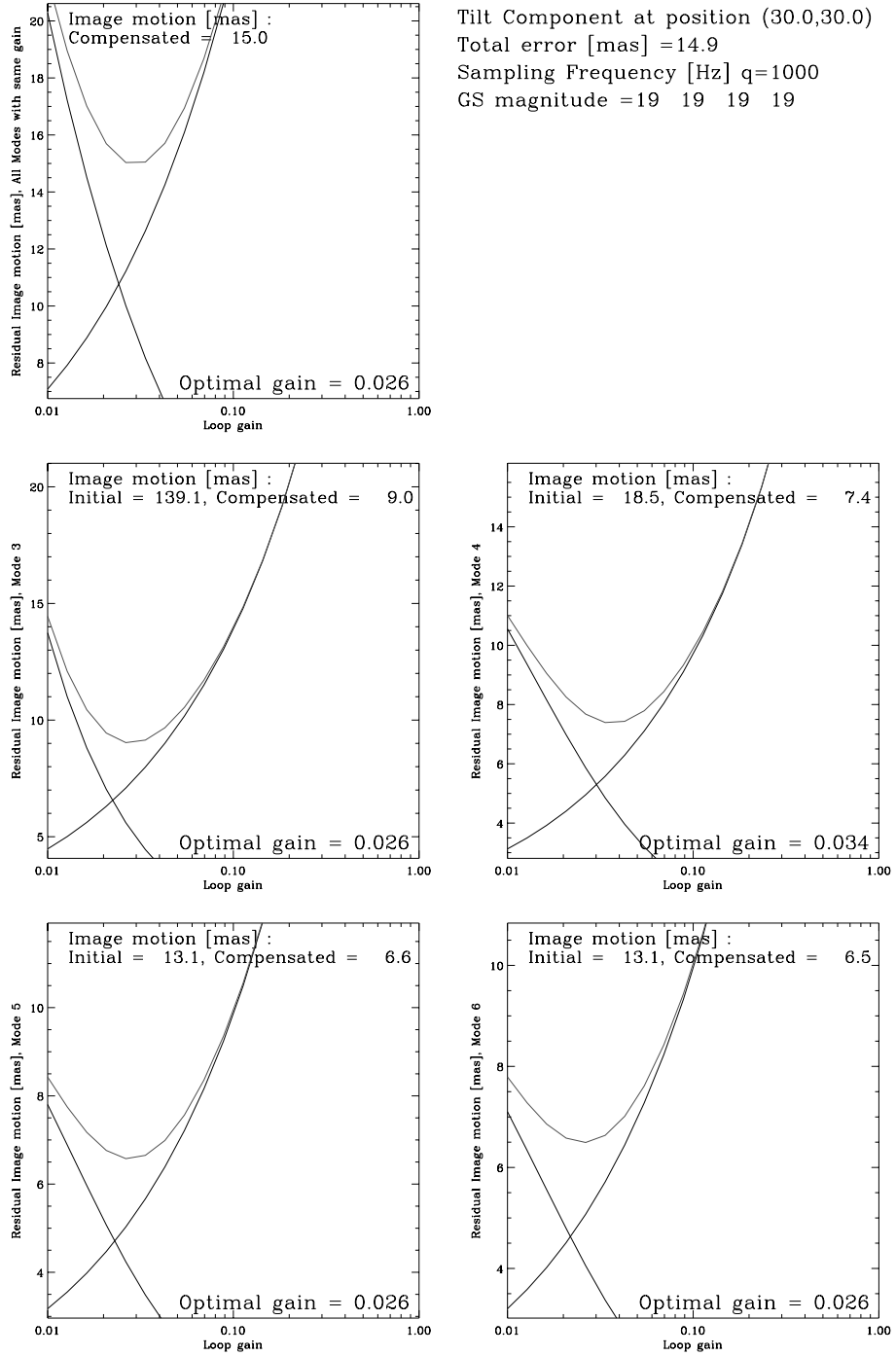


Figure 4: Same as figure 3 for the Tilt component

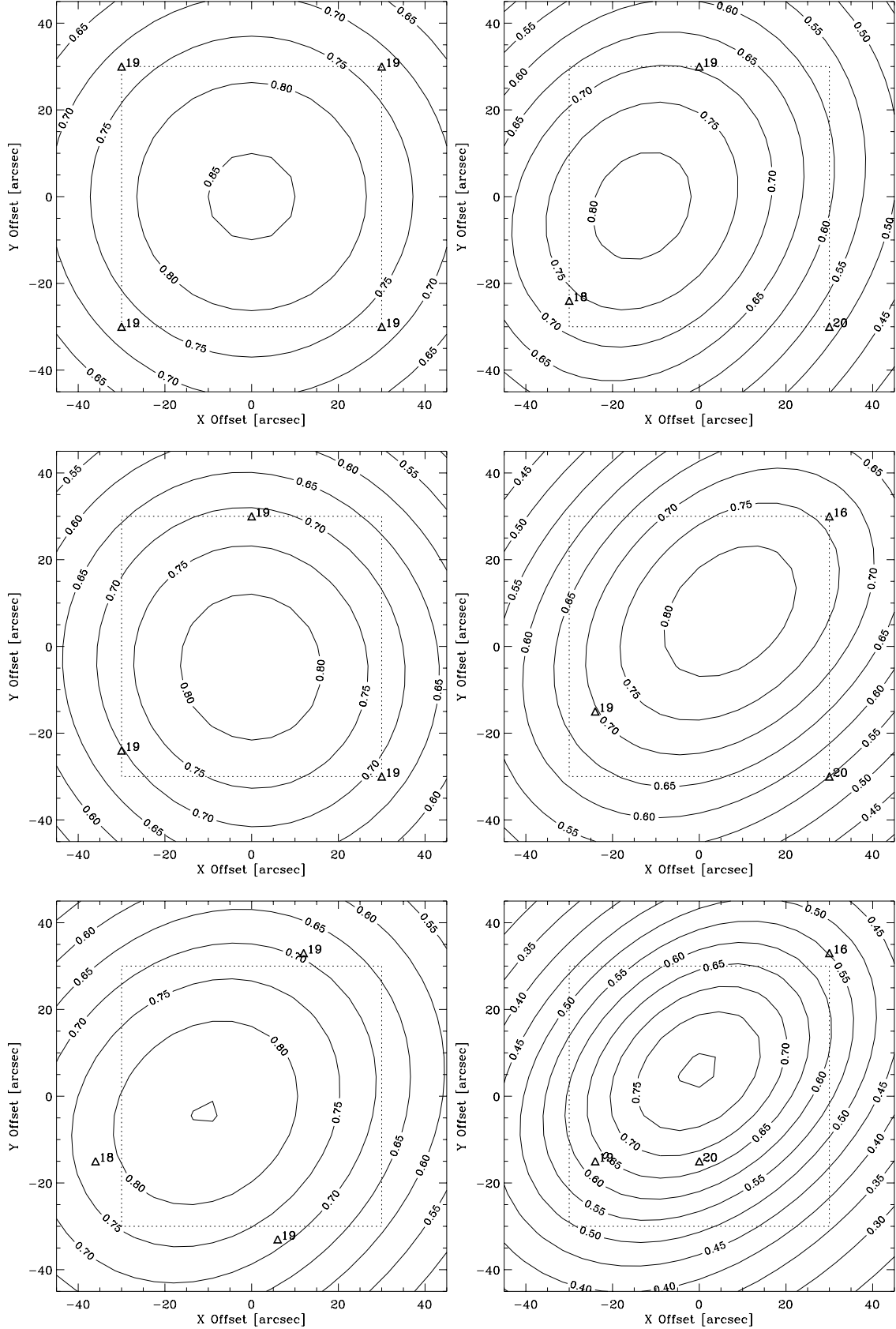


Figure 5: Strehl ratio loss due to the null modes versus field position for various guide star configuration and magnitudes, as plotted on the figures (triangle).

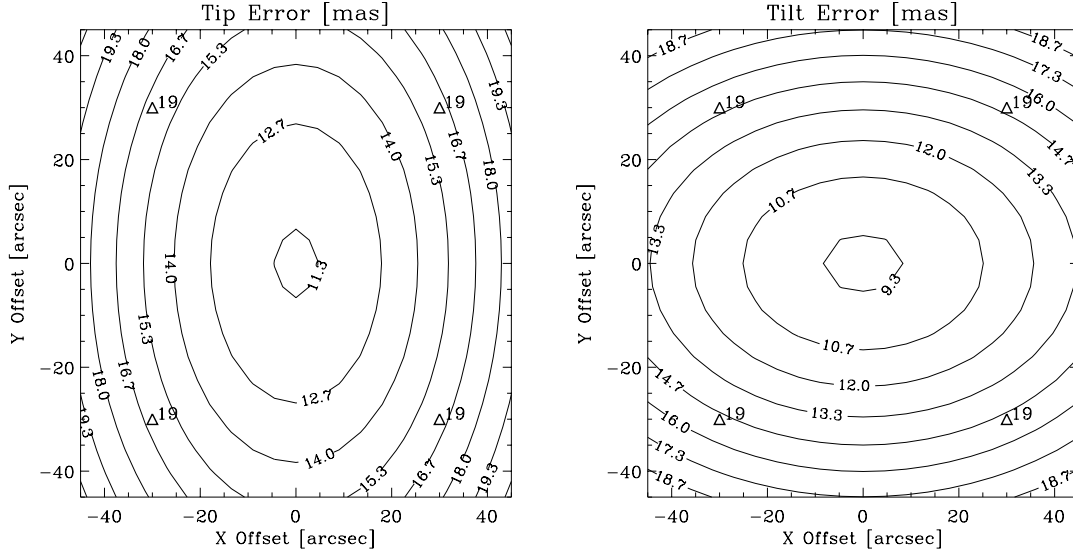


Figure 6: Same case as figure 3: Tip and Tilt error in milliarcsec versus field position

Once all these quantities are known, the tip and tilt compensated quadratic error, ξ^2 and ζ^2 can be evaluated. In Figure 3, the contribution of each null mode to ξ^2 (figure 4 for ζ^2) was computed versus the gain of the mode for a given point in the field (+30", +30"). The servo lag error and the noise terms (cf Eq 17) are computed separately. Fig 3 shows that, quite expectedly, the servo lag error (negative slope curve) on any given mode decreases as the loop gain increases (i.e. as the bandwidth goes up), while more noise gets propagated through the loop onto the compensated phase (positive slope curve). The optimal close-loop gain for any mode corresponds to the minimum of the sum of the servo lag error and noise terms, as shown on the figures. Because in the general case, the noise on the null mode may be correlated, and therefore the total error is *not* the straightforward addition of these four terms (cross terms in Eq 17), we have also computed the total tip and tilt error versus the loop gain. This is shown on top of figure 3 and 4 respectively, and assumes in this case an equal gain for all modes. For figure 3 and 4, computed for a four m=19 guide star system, and where the error is evaluated at (30,30) arcsec off-axis, the total tip error is 16.4 mas and the total tilt error is 15.0 mas. The error on axis is given by the tip and tilt modes only ($\theta_x = \theta_y = 0$ in Eq 17 and 18).

Eventually, the residual tip and tilt components were evaluated over 9 points in the one arcmin square field, and the optimal mode gains were chosen as the one that minimizes the average error over the evaluation points. When this is done, the errors can be computed versus the field position.

From the tip and tilt errors, one can also compute the Strehl ratio loss, which here is evaluated as:

$$\mathcal{S} = \sqrt{\frac{1}{1 + 2\sigma_x^2}} \times \sqrt{\frac{1}{1 + 2\sigma_y^2}} \quad (25)$$

where σ_x^2 and σ_y^2 are the tip and tilt phase variance, respectively. The Strehl loss at K band versus field is shown Figure 5, upper left, for the same case as figure 3 and 4.

Figure 6 presents the Tip (image motion along X) and Tilt error (along Y) errors versus the field position in milliarcsec. The asymmetry of each component is a signature of the fact that the compensated image motion is predominantly radial (both the turbulence residual *and* the noise).

Finally, Figure 6 presents also the K band Strehl loss for a sample of guide star configuration and magnitude.

This performance evaluation still lacks some aspects:

- The telescope shaking has not been included in the power spectra of the tip and tilt. Because the PSD of telescope shaking extend quite a way toward high frequencies, it may lower the performance. However, for telescope shaking, it is possible to make use of the peripheral WFS (P1 or P2). In that case, one may by compensating several minute off-axis indeed correct for the shaking but re-inject on axis the atmospheric tip-tilt at the sensing position. Because the 50% correlation angle of tip-tilt is of the order of several arcminutes, it should not significantly increase the amplitude of the atmospheric tip-tilt on axis.
- The use of smarter loop algorithms, such as for instance predictor, has to be investigated. It was shown earlier (e.g. Dessenne 1998) that predictors drastically improves performance in case of highly temporally structured perturbations, which the null modes are. This will be especially true when telescope shaking will be considered.